Basic Algebra II: Chapter 6

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Motivation: Extend the theory of modules to a bigger category.

- Initial object I: $\forall X \in ob\mathcal{C}$, $\exists!$ morphism $I \to X$.
- Terminal object T: $\forall X \in ob\mathcal{C}$, $\exists!$ morphism $X \to T$.
- Zero object 0: 0 is an initial object and terminal object.

- Zero hom $0: A \to B$:

  $A \longrightarrow 0 \longrightarrow B$ (unique)

- If $\mathcal{C}$ is a category with zero object, given $f: A \to B$

  $k: K \to A$ is a <u>kernel</u> of f if

  1) k is monic 2) $f \circ k = 0$ 3)

$$\begin{tikzcd} X \arrow[d, dashed, "\theta"'] \arrow[dr, "g"] \arrow[drr, "0"] & & \\ K \arrow[r, "k"] & A \arrow[r, "f"'] & B \end{tikzcd}$$

compatible with the usual case!

  $\pi: B \to C$ is a <u>cokernel</u> of f if

  1) π is epic 2) $\pi \circ f = 0$ 3)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[drr, "0"'] & B \arrow[r, "\pi"] \arrow[dr, "h"] & C \arrow[d, dashed, "\theta"] \\ & & Y \end{tikzcd}$$

Def. A category $\mathcal{C}$ is <u>additive</u> if

1. $\mathcal{C}$ has a zero object.
2. $\mathrm{Hom}(A,B)$ is an abelian gp $\forall A, B \in ob(\mathcal{C})$
3. Distribution Laws: $X \xrightarrow{a} A \overset{f}{\underset{g}{\rightrightarrows}} B \xrightarrow{b} Y$, then

   $(f+g)\circ a = fa + ga$ & $b \circ (f+g) = bf + bg$
4. $\mathcal{C}$ has finite product.

Def. If $\mathcal{C}$ and $\mathcal{D}$ are additive categories, a functor $T: \mathcal{C} \to \mathcal{D}$ is <u>additive</u> if $\forall f, g \in \mathrm{Hom}(A,B)$, $T(f+g) = Tf + Tg$

this is, $T: f \mapsto Tf$ is a group hom. $\mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(TA, TB)$

Def. A category $\mathcal{C}$ is an <u>abelian category</u> if it is an additive cat and

1) every mor has a kernel and cokernel
2) every monic is a kernel and every epic is a cokernel.

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \xrightarrow{u} C \Rightarrow f=g \qquad\qquad C \xrightarrow{\pi} A \overset{f}{\underset{g}{\rightrightarrows}} B \Rightarrow f=g$$

Ex. 1) ${}_R\mathrm{Mod}$ and Mod_R are abelian cats.

2) cat of torsion-free ab gps. is an additive cat.
but not an abelian cat. Consider $\mathbb{Z} \xrightarrow[\text{inclusion}]{} 2\mathbb{Z}$,
$\mathbb{Z}_2$ is its cokernal which is not ~~torsion~~-free.

Rmk. Any Abelian cat has finite inverse limit and finite direct limit.

pf. kernel exists $\Rightarrow$ equalizer exists, product exists $\Rightarrow$ inverse limit exists.

Def. A <u>complex</u> in an ab cat $\mathcal{C}$ is a seq of objs and mors

$$C = (C_\bullet, d_\bullet) = \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots$$

s.t. $d_n \circ d_{n+1} = 0$, $\forall n \in \mathbb{Z}$.

A <u>chain map</u> $f = f_\bullet : (C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$ is a seq of mors $f_n : C_n \to C'_n$
s.t. $d'_{n+1} \circ f_{n+1} = f_n \circ d_{n+1}$ $\forall n \in \mathbb{Z}$.

Rmk. The cat of all complexes in $\mathcal{C}$ is denoted by $\mathrm{Comp}(\mathcal{C})$, which is an ab cat.

Consider $\mathrm{Comp}({}_R\mathrm{Mod}) = {}_R\mathrm{Comp} \ni (C_\bullet, d_\bullet)$

- $Z_i(C) = \ker d_i$, $B_i(C) = \mathrm{Im}\, d_{i+1}$, $H_i(C) = Z_i(C)/B_i(C)$
  i-cycles. i-boundaries i-homology.

- $H_n : {}_R\mathrm{Comp} \to {}_R\mathrm{Mod}$ is an additive functor.

$$H_n : C \mapsto H_n(C) = Z_n / B_n .$$

$$f \mapsto H_n(f) = f_{n*} \quad \text{"induced map"}$$

$$\begin{tikzcd}
C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\
C'_{n+1} \arrow[r, "d'_{n+1}"] & C'_n \arrow[r, "d'_n"] & C'_{n-1}
\end{tikzcd}$$

Rmk. C is exact $\Leftrightarrow H_n(C) = 0$, $\forall n \in \mathbb{Z}$.

$$\begin{tikzcd}
H_n(C) \ni cls(z) \arrow[d, "f_{n*}"] \\
H_n(C') \ni cls(f_n(z))
\end{tikzcd}$$

- We can rewrite a complex

$$\cdots \to C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \to \cdots \quad \text{"decreasing"}$$

to

$$\cdots \to C^{-(i+1)} \xrightarrow{d^{-(i+1)}} C^{-i} \xrightarrow{d^{-i}} C^{-(i-1)} \to \cdots \quad \text{cochain complex}$$

this is

$$\cdots \to C^{i-1} \xrightarrow{d^{i-1}} C^{i} \xrightarrow{d^{i}} C^{i+1} \to \cdots \quad \text{"increasing"}$$

$$C'_n \xrightarrow{\alpha_n} C_n \xrightarrow{\beta_n} C''_n$$
$$\Downarrow$$
$$H_n(C') \xrightarrow{\alpha_{n*}} H_n(C) \xrightarrow{\beta_{n*}} H_n(C'')$$

- A cochain complex C.

$Z^i(C) = \ker d^i$, $B^i(C) = \mathrm{Im}\, d^{i-1}$, $H^i(C) = Z^i(C)/B^i(C)$

i-cocycles, i-coboundaries, i-cohomology

Def. $0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to 0$ is SES of complexes

$\Leftrightarrow 0 \to C'_n \xrightarrow{\alpha_n} C_n \xrightarrow{\beta_n} C''_n \to 0$ is SES of $\mathcal{C}$, $\forall n \in \mathbb{Z}$

Thm. $0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to 0$ SES in ${}_R\mathrm{Comp}$. then there exists a morphism $\partial_n : H_n(C'') \to H_{n-1}(C') : cls(z'') \mapsto cls(\alpha_{n-1}^{-1} d_n \beta_n^{-1}(z''))$ "Connecting hom" such that $\to H_n(C') \xrightarrow{\alpha_{n*}} H_n(C) \xrightarrow{\beta_{n*}} H_n(C'') \xrightarrow{\partial_n} H_{n-1}(C') \xrightarrow{\alpha_{n-1*}} H_{n-1}(C) \to \cdots$ exact.

"Long Exact Seq"

Cor. (Snake Lemma) A commutative diagram with exact rows:

$$\begin{tikzcd}
 & \Sigma' f & \Sigma' g & \Sigma' h & \\
0 \arrow[r] & A' \arrow[r] \arrow[d, "f"] & A \arrow[r] \arrow[d, "g"] & A'' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0
\end{tikzcd}$$

implies $0 \to \ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h \to 0$

$0 \to H_1(\Sigma' f) \to H_1(\Sigma' g) \to H_1(\Sigma' h) \to H_0(\Sigma' f) \to H_0(\Sigma' g) \to H_0(\Sigma' h) \to 0$

Thm.

$$\begin{tikzcd} 0 \arrow[r] & C' \arrow[r, "i"] \arrow[d, "f"'] & C \arrow[r, "p"] \arrow[d, "g"'] & C'' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & D' \arrow[r, "j"'] & D \arrow[r, "q"'] & D'' \arrow[r] & 0 \end{tikzcd}$$

commutative and exact in ${}_R\mathrm{Comp}$

then

$$\begin{tikzcd} \arrow[r] & H_n(C') \arrow[r, "i_{n*}"] \arrow[d, "f_{n*}"'] & H_n(C) \arrow[r, "p_{n*}"] \arrow[d, "g_{n*}"'] & H_n(C'') \arrow[r, "\partial_n"] \arrow[d, "h_{n*}"'] & H_{n-1}(C') \arrow[r] \arrow[d, "f_{n-1*}"] & \cdots \\ \arrow[r] & H_n(D') \arrow[r, "j_{n*}"'] & H_n(D) \arrow[r, "q_{n*}"'] & H_n(D'') \arrow[r, "\partial_n'"'] & H_{n-1}(D') \arrow[r] & \cdots \end{tikzcd}$$

commutative and exact.

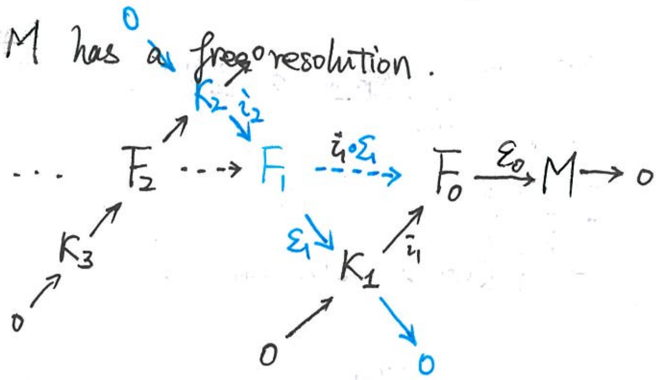
Def. (C, ε) is a <u>resolution</u> of M if

augmentation

$$\to C_n \xrightarrow{d_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{d_1} C_0 \xrightarrow{\varepsilon} M \to 0$$

is exact.

- $H_0(C) = C_0/d_1C_1 = C_0/\ker\varepsilon = M$
- C_i is projective $\forall i \in \mathbb{N}$, then (C, ε) is a projective resolution.
- Every R-module M has a free resolution.



Thm. (Comparison Theorem) $f: A \to A'$ in an ab cat $\mathcal{C}$, P projective resolution of A, P' resolution of A', then $\exists\, \check{f}$ s.t. following diagram commutes

$$\begin{tikzcd} \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, dashed, "\check{f}_1"] & P_0 \arrow[r] \arrow[d, dashed, "\check{f}_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\ \cdots \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r] & A' \arrow[r] & 0 \end{tikzcd}$$

Pf. By def of projective.

Also proved by def of projectivity.

Rmk. $\check{f}$ is not unique, but they are homotopic.

Def. $(C,d) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} (C',d')$. then α is <u>homotopic</u> to β if there exists $s=\{s_i : C_i \to C'_{i+1}\}$

s.t. $\alpha_i - \beta_i = d'_{i+1} s_i + s_{i-1} d$.

$(\alpha \simeq \beta)$

$$\begin{tikzcd}
\arrow[r] & C_{i+1} \arrow[r, "d_{i+1}"] \arrow[d] & C_i \arrow[r, "d_i"] \arrow[d] \arrow[dl, dashed, "s_i"'] & C_{i-1} \arrow[r] \arrow[d] \arrow[dl, dashed, "s_{i-1}"'] & {} \\
\arrow[r] & C'_{i+1} \arrow[r] & C'_i \arrow[r] & C'_{i-1} \arrow[r] & {}
\end{tikzcd}$$

Rmk. There is also a "injective" version of "Comparision Theorem."

Thm. If $\alpha \simeq \beta : (C,d) \to (C',d')$, then $\alpha_{n*} = \beta_{n*} : H_n(C) \to H_n(C')$. $\forall n \in \mathbb{Z}$.

Homotopic chain maps induce the same morphism in homology.

Pf. $\forall z \in Z_n(C)$, $(\alpha_n - \beta_n)z = (d'_{n+1} s_n + s_{n-1} d_n) z = d'_{n+1}(s_n(z)) \in B_n(C')$

$\alpha_{n*}(\mathrm{cls}(z)) = \mathrm{cls}(\alpha_n z) = \mathrm{cls}((\alpha_n - \beta_n + \beta_n) z) = \mathrm{cls}(\beta_n z) = \beta_{n*}$

• Left Derived Functors

Consider an additive covariant functor F between two ab cats, $\mathcal{C}$ and $\mathcal{D}$.

$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0$ < Take $f : A \to A'$ in $\mathcal{C}$.

$\downarrow \check{f} \qquad \downarrow f$

$\cdots \to P'_2 \to P'_1 \to P'_0 \to A' \to 0$

$\Downarrow$ F additive covariant

$\cdots \to FP_2 \to FP_1 \to FP_0 \to FA \to 0$

$\downarrow F\check{f} \qquad \downarrow Ff$

$\cdots \to FP'_2 \to FP'_1 \to FP'_0 \to FA' \to 0$

$\Downarrow$ $F\check{f} : FP_A \to FP_{A'}$ chain map

$(F\check{f})_{n*} = H_n(F\check{f}) : H_n(FP_A) \to H_n(FP_{A'})$

According to the process, $H_n(F\check{f})$ seems to depend on the choice of $\check{f}$ & P_A. But it is not true.

- Let $\check{f}$ and $\check{g}$ be the chain map over f

  $\check{f} \simeq \check{g} \Rightarrow F\check{f} \simeq F\check{g} \Rightarrow H_n(F\check{f}) = H_n(F\check{g})$

- Introduction to Homological Algebra, Rotman. proposition 6.1

So we def $L_nF : \mathcal{C} \to \mathcal{D}$.

$L_nF(A) = H_n(FP_A)$, $L_nF(f) = H_n(F\check{f})$

<u>Thm</u>. $\mathcal{C}, \mathcal{D}$ ab cats with enough projectives. $F: \mathcal{C} \to \mathcal{D}$ additive covariant functor.

$0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0$ SES $\Rightarrow \cdots \to (L_nF)A' \xrightarrow{(L_nF)i} (L_nF)A \xrightarrow{(L_nF)p} (L_nF)A'' \xrightarrow{\partial_n}$

$(L_{n-1}F)A' \xrightarrow{(L_{n-1}F)i} (L_{n-1}F)A \to \cdots$ LES

pf.

SES $0 \to A' \to A \to A'' \to 0$

Horseshoe Lemma

SES $0 \to P_{A'} \to P_A \to P_{A''} \to 0$

projective: split SES

additive functor: preserve split SES.

SES $0 \to FP_{A'} \to FP_A \to FP_{A''} \to 0$

Long exact seq

$\cdots \to (L_nF)A' \to (L_nF)A \to (L_nF)A'' \xrightarrow{\partial_n} (L_{n-1}F)A' \to (L_{n-1}F)A \to \cdots$